

# LOCALIZATION OF INJECTIVE MODULES OVER ARITHMETICAL RINGS

FRANÇOIS COUCHOT

**ABSTRACT.** It is proved that localizations of injective  $R$ -modules of finite Goldie dimension are injective if  $R$  is an arithmetical ring satisfying the following condition: for every maximal ideal  $P$ ,  $R_P$  is either coherent or not semicoherent. If, in addition, each finitely generated  $R$ -module has finite Goldie dimension, then localizations of finitely injective  $R$ -modules are finitely injective too. Moreover, if  $R$  is a Prüfer domain of finite character, localizations of injective  $R$ -modules are injective.

This is a sequel and a complement of (Couchot, 2006). If  $R$  is a noetherian or hereditary ring, it is well known that localizations of injective  $R$ -modules are injective. By (Couchot, 2006, Corollary 8) this property holds if  $R$  is a h-local Prüfer domain. However (Couchot, 2006, Example 1) shows that this result is not generally true. E. C. Dade was probably the first to study localizations of injective modules. By (Dade, 1981, Theorem 25), there exist a ring  $R$ , a multiplicative subset  $S$  and an injective module  $G$  such that  $S^{-1}G$  is not injective. In this example we can choose  $R$  to be a coherent domain.

The aim of this paper is to study localizations of injective modules over arithmetical rings. We deduce from (Couchot, 2006, Theorem 3) the two following results: any localization of an injective  $R$ -module of finite Goldie dimension is injective if and only if any localization at a maximal ideal of  $R$  is either coherent or non-semicoherent (Theorem 5) and each localization of any injective module over a Prüfer domain of finite character is injective (Theorem 10). Moreover, if any localization at a maximal ideal of  $R$  is either coherent or non-semicoherent, and if each finitely generated  $R$ -module has a finite Goldie dimension, then each localization of any finitely injective  $R$ -module is finitely injective.

In this paper all rings are associative and commutative with unity and all modules are unital. A module is said to be *uniserial* if its submodules are linearly ordered by inclusion. A ring  $R$  is a *valuation ring* if it is uniserial as  $R$ -module and  $R$  is *arithmetical* if  $R_P$  is a valuation ring for every maximal ideal  $P$ . An arithmetical domain  $R$  is said to be *Prüfer*. We say that a module  $M$  is of *Goldie dimension*  $n$  if and only if its injective hull  $E(M)$  is a direct sum of  $n$  indecomposable injective modules. We say that a domain  $R$  is of *finite character* if every non-zero element is contained in finitely many maximal ideals.

As in (Ramamurthi and Rangaswamy, 1973), a module  $M$  over a ring  $R$  is said to be *finitely injective* if every homomorphism  $f : A \rightarrow M$  extends to  $B$  whenever  $A$  is a finitely generated submodule of an arbitrary  $R$ -module  $B$ .

---

2000 *Mathematics Subject Classification.* Primary 13F05, 13C11.

*Key words and phrases.* arithmetical ring, semicoherent ring, injective module, FP-injective module, finitely injective module, Goldie dimension, valuation ring, Prüfer domain, finite character.

As in (Matlis, 1985) a ring  $R$  is said to be *semicoherent* if  $\text{Hom}_R(E, F)$  is a submodule of a flat  $R$ -module for any pair of injective  $R$ -modules  $E, F$ . An  $R$ -module  $E$  is *FP-injective* if  $\text{Ext}_R^1(F, E) = 0$  for any finitely presented  $R$ -module  $F$ , and  $R$  is *self FP-injective* if  $R$  is FP-injective as  $R$ -module. We recall that a module  $E$  is FP-injective if and only if it is a pure submodule of every overmodule. If each injective  $R$ -module is flat we say that  $R$  is an *IF-ring*. By (Colby, 1975, Theorem 2),  $R$  is an IF-ring if and only if it is coherent and self FP-injective.

We begin by some results on semicoherent rings.

**Proposition 1.** *Let  $R$  be a self FP-injective ring. Then  $R$  is coherent if and only if it is semicoherent.*

*Proof.* If  $R$  is coherent then, by (Fuchs and Salce, 2001, Theorem XIII.6.4(b)),  $\text{Hom}_R(E, F)$  is flat for any pair of injective modules  $E, F$ ; so,  $R$  is semicoherent. Conversely, let  $E$  be the injective hull of  $R$ . Since  $R$  is a pure submodule of  $E$ , then, for each injective  $R$ -module  $F$ , the following sequence is exact:

$$0 \rightarrow \text{Hom}_R(F \otimes_R E/R, F) \rightarrow \text{Hom}_R(F \otimes_R E, F) \rightarrow \text{Hom}_R(F \otimes_R R, F) \rightarrow 0.$$

By using the natural isomorphisms  $\text{Hom}_R(F \otimes_R B, F) \cong \text{Hom}_R(F, \text{Hom}_R(B, F))$  and  $F \cong \text{Hom}_R(R, F)$  we get the following exact sequence:

$$0 \rightarrow \text{Hom}_R(F, \text{Hom}_R(E/R, F)) \rightarrow \text{Hom}_R(F, \text{Hom}_R(E, F)) \rightarrow \text{Hom}_R(F, F) \rightarrow 0.$$

So, the identity map on  $F$  is the image of an element of  $\text{Hom}_R(F, \text{Hom}_R(E, F))$ . Consequently the following sequence splits:

$$0 \rightarrow \text{Hom}_R(E/R, F) \rightarrow \text{Hom}_R(E, F) \rightarrow F \rightarrow 0.$$

It follows that  $F$  is a summand of a flat module. So,  $R$  is an IF-ring.  $\square$

**Corollary 2.** *Let  $R$  be a ring such that its ring of quotients  $Q$  is self FP-injective. Then  $R$  is semicoherent if and only if  $Q$  is coherent.*

*Proof.* If  $R$  is semicoherent, then so is  $Q$  by (Matlis, 1985, Proposition 1.2). From Proposition 1 we deduce that  $Q$  is coherent. Conversely, let  $E$  and  $F$  be injective  $R$ -modules. It is easy to check that the multiplication by a regular element of  $R$  in  $\text{Hom}_R(E, F)$  is injective. So,  $\text{Hom}_R(E, F)$  is a submodule of the injective hull of  $Q \otimes_R \text{Hom}_R(E, F)$  which is flat over  $Q$  and  $R$  because  $Q$  is an IF-ring.  $\square$

From Corollary 2 and (Couchot, 2003, Theorem II.11) we deduce the following:

**Corollary 3.** *Let  $R$  be a valuation ring. Denote by  $Z$  its subset of zero divisors which is a prime ideal. Assume that  $Z \neq 0$ . Then the following conditions are equivalent:*

- (1)  $R$  is semicoherent;
- (2)  $R_Z$  is an IF-ring;
- (3)  $Z$  is not a flat  $R$ -module.

From Corollary 3 and (Couchot, 2006, Theorem 3) we deduce the following:

**Corollary 4.** *Let  $R$  be a valuation ring of maximal ideal  $P$ . Then the following conditions are equivalent:*

- (1)  $R$  is either coherent or non-semicoherent;

- (2) for each multiplicative subset  $S$  of  $R$  and for each injective  $R$ -module  $E$ ,  $S^{-1}E$  is injective;
- (3) for each multiplicative subset  $S$  of  $R$  and for each FP-injective  $R$ -module  $E$ ,  $S^{-1}E$  is FP-injective;
- (4)  $(E_R(R/P))_Z$  is FP-injective.

*Proof.* (1)  $\Rightarrow$  (2). Since  $R$  is a valuation ring,  $R \setminus S$  is a prime ideal. If  $R$  is coherent then either  $Z = 0$  or  $Z = P$ . In the first case  $Z$  is flat and in the second  $E$  is flat. So we conclude by (Couchot, 2006, Theorem 3). If  $R$  is not semicoherent then  $Z$  is flat. We conclude in the same way.

(2)  $\Rightarrow$  (3).  $E$  is a pure submodule of its injective hull  $H$ . Then  $S^{-1}E$  is a pure submodule of  $S^{-1}H$  which is injective by (2). So,  $S^{-1}E$  is FP-injective.

(3)  $\Rightarrow$  (4) is obvious.

(4)  $\Rightarrow$  (1). Suppose that  $Z$  is not flat. If  $R$  is not coherent, then, by (Couchot, 2003, Theorem II.11),  $Z \neq 0$  and  $Z \neq P$ , and  $R$  is not self FP-injective. Let  $E = E_R(R/P)$ . By (Couchot, 1982, Proposition 2.4)  $E$  is not flat. Now, we do as in the last part of the proof of (Couchot, 2006, Theorem 3) to show that  $E_Z$  is not FP-injective. This contradicts (4). The proof is now complete.  $\square$

**Theorem 5.** For any arithmetical ring  $R$  the following conditions are equivalent:

- (1) for each maximal ideal  $P$ ,  $R_P$  is either coherent or non-semicoherent;
- (2) for each multiplicative subset  $S$  and for each injective  $R$ -module  $G$  of finite Goldie dimension,  $S^{-1}G$  is injective;
- (3) for each multiplicative subset  $S$  and for each FP-injective  $R$ -module  $G$  of finite Goldie dimension,  $S^{-1}G$  is FP-injective;
- (4) for each maximal ideal  $P$ ,  $Q(R_P) \otimes_R E_R(R/P)$  is FP-injective, where  $Q(R_P)$  is the ring of fractions of  $R_P$ .

*Proof.* (1)  $\Rightarrow$  (2).  $G$  is a finite direct sum of indecomposable injective modules. We may assume that  $G$  is indecomposable. Since  $\text{End}_R G$  is local, there exists a maximal ideal  $P$  such that  $G$  is a module over  $R_P$ . If  $S'$  is the multiplicative subset of  $R_P$  generated by  $S$ , then  $S'^{-1}G = S^{-1}G$ . We conclude that  $S^{-1}G$  is injective by Corollary 4.

We show (2)  $\Rightarrow$  (3) as in the proof of Corollary 4, and (3)  $\Rightarrow$  (4) is obvious.

(4)  $\Rightarrow$  (1) is an immediate consequence of Corollary 4.  $\square$

**Remark 6.** If  $R$  is an arithmetical ring which is coherent or reduced, then  $R$  satisfies the conditions of Theorem 5.

**Corollary 7.** Let  $R$  be an arithmetical ring satisfying the following two conditions:

- (a) for each maximal ideal  $P$ ,  $R_P$  is either coherent or non-semicoherent;
- (b) every finitely generated  $R$ -module has a finite Goldie dimension.

Then, for each multiplicative subset  $S$  and for each finitely injective (respectively FP-injective)  $R$ -module  $G$ ,  $S^{-1}G$  is finitely injective (respectively FP-injective).

Moreover, if  $R_P$  is coherent for each maximal ideal  $P$  then  $R$  is coherent too.

*Proof.* Let  $M$  be a finitely generated  $S^{-1}R$ -submodule of  $S^{-1}G$ . There exists a finitely generated submodule  $M'$  of  $G$  such that  $M = S^{-1}M'$ . If  $G$  is finitely injective, by (Ramamurthi and Rangaswamy, 1973, Proposition 3.3) it contains an injective hull  $E$  of  $M'$ . Then  $E$  has finite Goldie dimension. By Theorem 5  $S^{-1}E$  is injective. It contains  $M$  and it is contained in  $S^{-1}G$ . By using again

(Ramamurthi and Rangaswamy, 1973, Proposition 3.3) we conclude that  $S^{-1}G$  is finitely injective.

If  $G$  is FP-injective, it is a pure submodule of its injective hull  $H$ . Then  $S^{-1}G$  is a pure submodule of  $S^{-1}H$  which is finitely injective. So,  $S^{-1}G$  is FP-injective.

The last assertion is an immediate consequence of (Couchot, 1982, Théorème 1.4).  $\square$

**Remark 8.** If  $R$  is an arithmetical ring satisfying the condition (b) of Corollary 7 then  $\text{Min } R/A$  is finite for each ideal  $A$ : we may assume that  $A = 0$  and  $R$  is reduced; its total ring of quotient is Von Neumann regular by (Lambek, 1966, Proposition 2 p. 106) and semisimple by (Lambek, 1966, Proposition 2 p. 103); it follows that  $\text{Min } R$  is finite. However, the converse doesn't hold. For instance, let  $R = \{ \begin{pmatrix} d & e \\ 0 & d \end{pmatrix} \mid d \in \mathbb{Z}, e \in \mathbb{Q}/\mathbb{Z} \}$  be the trivial extension of  $\mathbb{Z}$  by  $\mathbb{Q}/\mathbb{Z}$ . Then  $N = \{ \begin{pmatrix} 0 & e \\ 0 & 0 \end{pmatrix} \mid e \in \mathbb{Q}/\mathbb{Z} \}$  is the only minimal prime. For each prime integer  $p$ , the localization  $R_{(p)}$  is the trivial extension of  $\mathbb{Z}_{(p)}$  by  $\mathbb{Q}/\mathbb{Z}_{(p)}$ . So it is a valuation ring. Consequently  $R$  is arithmetical. But  $N \cong \mathbb{Q}/\mathbb{Z}$  is an infinite direct sum, whence  $R$  is not a module of finite Goldie dimension.

By (Couchot, 2006, Corollary 8), if  $R$  is a h-local Prüfer domain, all localizations of injective  $R$ -modules are injective. Now, we extend this result to each Prüfer domain of finite character. A such ring satisfies condition(b) of Corollary 7. But  $\mathbb{Z} + X\mathbb{Q}[[X]]$  is an example showing that the converse doesn't hold.

**Lemma 9.** *Let  $R$  be a Prüfer domain of finite character. For each maximal ideal  $P$ , let  $F_{(P)}$  be an injective  $R_P$ -module and let  $F = \prod_{P \in \text{Max } R} F_{(P)}$ . Then  $S^{-1}F$  is injective for every multiplicative subset  $S$  of  $R$ .*

*Proof.* Let  $T_{(P)}$  be the torsion submodule of  $F_{(P)}$ , let  $G_{(P)} = F_{(P)}/T_{(P)}$ , let  $T = \prod_{P \in \text{Max } R} T_{(P)}$  and let  $G = \prod_{P \in \text{Max } R} G_{(P)}$ . Then  $G$  is torsion-free and  $F \cong T \oplus G$ . It is obvious that  $S^{-1}G$  is injective. Let  $T' = \oplus_{P \in \text{Max } R} T_{(P)}$ . Since  $R$  has finite character, it is easy to check that  $T'$  is the torsion submodule of  $T$ . So,  $T'$  is injective and  $S^{-1}(T/T')$  is injective. For each maximal ideal  $P$ ,  $S^{-1}T_{(P)}$  is injective by (Couchot, 2006, Theorem 3). Since  $S^{-1}T'$  is the torsion submodule of  $\prod_{P \in \text{Max } R} S^{-1}T_{(P)}$ , we successively deduce the injectivity of  $S^{-1}T'$  and  $S^{-1}T$ .  $\square$

**Theorem 10.** *Let  $R$  be a Prüfer domain of finite character. Then, for each injective module  $G$ ,  $S^{-1}G$  is injective for every multiplicative subset  $S$  of  $R$ .*

*Proof.* Let  $E = \prod_{P \in \text{Max } R} E_R(R/P)$  and let  $F = \text{Hom}_R(\text{Hom}_R(G, E), E)$ . Then  $E$  is an injective cogenerator and  $G$  is isomorphic to a summand of  $F$ . Since  $R$  is coherent,  $\text{Hom}_R(G, E)$  is flat by (Fuchs and Salce, 2001, Theorem XIII.6.4(b)). Thus  $F$  is injective. We put  $F_{(P)} = \text{Hom}_R(\text{Hom}_R(G, E), E_R(R/P))$ . Then  $F_{(P)}$  is an injective  $R_P$ -module and  $F \cong \prod_{P \in \text{Max } R} F_{(P)}$ . By Lemma 9  $S^{-1}F$  is injective. We conclude that  $S^{-1}G$  is injective too.  $\square$

**Corollary 11.** *Let  $R$  be a semilocal Prüfer domain. Then, for each injective module  $G$ ,  $S^{-1}G$  is injective for every multiplicative subset  $S$  of  $R$ .*

The following example shows that the finite character is not a necessary condition in order that localizations of injective modules at multiplicative subsets are still injective.

**Example 12.** Let  $R$  be the ring defined in (Hutchins, 1981, Example 39). Then  $R$  is a Prüfer domain which is not of finite character. But, since  $R$  is the union of a countable family of principal ideal subrings, it is easy to check that, for any multiplicative subset  $S$ ,  $R$  satisfies (Dade, 1981, Condition 14). So, for each injective module  $G$ ,  $S^{-1}G$  is injective by (Dade, 1981, Theorem 15).

Here another example communicated to me by L. Salce. Take  $R$  constructed as in Chapter III, Example 5.5 of (Fuchs and Salce, 2001), which is a classical example by Heinzer-Ohm of almost Dedekind domain not of finite character. If you start with a countable field  $K$ , then  $R$  is countable, hence conditions (14a) and (14c) of (Dade, 1981) are satisfied. Condition (14b) must be checked only for  $I$  principal ideal, and it is easy to see that it holds true.

Consequently, the following question is unsolved:

**Open question:** characterize the Prüfer domains such that localizations of injective at multiplicative subsets are still injective.

#### REFERENCES

Colby, R. (1975). Flat injective modules. *J. Algebra*, 35:239–252.

Couchot, F. (1982). Exemples d’anneaux auto fp-injectifs. *Comm. Algebra*, 10(4):339–360.

Couchot, F. (2003). The  $\lambda$ -dimension of commutative arithmetic rings. *Comm. Algebra*, 31(7):3143–3158.

Couchot, F. (2006). Localization of injective modules over valuations rings. *Proc. Amer. Math. Soc.*, 134(4):1013–1017.

Dade, E. (1981). Localization of injective modules. *J. Algebra*, 69:416–425.

Fuchs, L. and Salce, L. (2001). *Modules over Non-Noetherian Domains*. Number 84 in Mathematical Surveys and Monographs. American Mathematical Society, Providence.

Hutchins, H. (1981). *Examples of commutative rings*. Polygonal Publishing House.

Lambek, J. (1966). *Lectures on rings and modules*. Blaisdell. Waltham.

Matlis, E. (1985). Commutative semicoherent and semiregular rings. *J. Algebra*, 95(2):343–372.

Ramamurthi, V. and Rangaswamy, K. (1973). On finitely injective modules. *J. Aust. Math. Soc.*, XVI(2):239–248.

LABORATOIRE DE MATHÉMATIQUES NICOLAS ORESME, CNRS UMR 6139, DÉPARTEMENT DE MATHÉMATIQUES ET MÉCANIQUE, 14032 CAEN CEDEX, FRANCE  
*E-mail address:* couchot@math.unicaen.fr